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False time-reversal violation and energy level statistics: the role of anti-unitary symmetry

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Abstract. We extend the classification of symmetries necessary to predict the universality class of spectral fluctuations of quantal systems whose classical motion is chaotic, by explaining that a system with neither time-reversal symmetry (T) nor geometric symmetry may display the spectral statistics of the Gaussian orthogonal ensemble (GOE), rather than those of the Gaussian unitary ensemble (GUE), provided it possesses instead some combination of symmetries which includes T. Such combinations constitute invariance under anti-unitary transformations (whose classical analogue we call anticanonical). For a particle in a magnetic field B plus scalar potential V, an example is TS_x where S_x is a mirror reflection under which B and V are invariant. We illustrate this numerically for a single flux line in a hard-walled enclosure (Aharonov-Bohm quantum billiards), which also provides an example of an anti-unitary symmetry of non-geometrical origin; the spectral fluctuations are, as predicted, GOE rather than GUE.

1. Introduction

It is now firmly established that the local statistics of energy levels (spectral fluctuations) of classically chaotic quantum systems are the same as the statistics of large random matrices. (Porter (1965) gives a collection of papers about random-matrix theory; computations indicating the applicability of this theory to quantal systems whose classical motion is chaotic are reviewed by Bohigas and Giannoni (1984); the reasons for this applicability are discussed in theoretical papers by Pechukas (1983), Berry (1985) and Yukawa (1985).)

Spectral fluctuations fall into two universality classes, represented by different matrix ensembles; both apply only to systems without purely geometric symmetry, or if such symmetry is present, to level sequences of states in the same symmetry class. The first spectral universality class is that of the Gaussian orthogonal ensemble (GOE) of random real symmetric matrices; it applies to systems possessing time-reversal symmetry (T), and clear numerical evidence was presented by Bohigas *et al* (1984). The second spectral universality class is that of the Gaussian unitary ensemble (GUE) of random complex Hermitian matrices; it has been the conventional wisdom that this applies to systems without T, and clear numerical evidence was presented by Berry and Robnik (1985) and Seligman and Verbaarschot (1985).

Our purpose here is to point out that this conventional wisdom is not always correct. We demonstrate theoretically, and with examples, that there exist systems (indeed commonly encountered ones) whose dynamics are invariant neither under the T operation nor under any geometric symmetry operation, but for which non-trivial representations can be found in which the Hamiltonian matrix elements are real, so that GOE statistics apply, rather than GUE statistics. This phenomenon, which we call false T violation, arises when the system possesses invariance under a *combination* of T and some other symmetry (which may be geometric—for example, a reflection—but may be of a dynamical nature). In quantum mechanics such symmetries are *anti-unitary*, and in § 2 we explain how they can give rise to real Hamiltonians. The natural example, with which we illustrate these ideas, is that of a charged particle in external magnetic and scalar fields; we describe this in § 3, where we also introduce the classical analogues of anti-unitary symmetries, which we call *anticanonical.* In § 4, we present numerical results for the case where the magnetic field is concentrated into a single flux line and the scalar field is the repulsive potential of a hard-walled enclosure (the Aharonov-Bohm quantum billiard, see Berry and Robnik (1986)), showing how anti-unitary symmetry can give rise to GOE spectral statistics even though T is violated.

In this as in all applications of random-matrix theory, we are predicting that the spectral statistics of an individual system will be the same as those averaged over an ensemble (GOE or GUE) of systems. The status of such predictions is discussed in § 5.

To avoid confusion, we emphasise that here we are concerned only with systems whose classical motion is chaotic, that is, ergodic with all closed orbits unstable. Different spectral statistics occur for completely integrable systems (Berry and Tabor 1977) or for the generic case of systems whose motion is between the integrable and chaotic extremes (Robnik 1984, Meyer *et al* 1984, Seligman *et al* 1984, 1985, Berry and Robnik 1984).

2. Anti-unitary symmetry and real representations

Invariance under time reversal T is a particular case of invariance of the Hamiltonian operator $\hat{H}$ under the action of an *anti-unitary operator* $\hat{A}$. The general theory of such operators was given by Wigner (1959), Dyson (1962), Bargmann (1964) and Porter (1965, pp 2–87); here we repeat only the essential facts.

Any $\hat{A}$ can be expressed as the successive application of the operator of complex conjugation $\hat{K}$, and a unitary operator $\hat{U}$, i.e.

$$\hat{A} = \hat{U}\hat{K}. \tag{1}$$

It follows that $\hat{A}$ is *anti-linear*: for any states $|\psi\rangle$, $|\phi\rangle$ and any complex numbers a, b

$$\hat{A}(a|\psi\rangle + b|\phi\rangle) = a^*\hat{A}|\psi\rangle + b^*\hat{A}|\phi\rangle. \tag{2}$$

Moreover, $\hat{A}$ preserves the transition probability between any two states, by converting the transition amplitude to its complex conjugate, that is

$$\langle\hat{A}\phi|\hat{A}\psi\rangle = \langle\phi|\psi\rangle^*. \tag{3}$$

This property defines anti-unitarity; with an appropriate definition of the anti-adjoint it can be written $\hat{A}^\dagger\hat{A} = 1$. Under $\hat{A}$, any dynamical operator $\hat{Q}$ transforms to $\hat{Q}_A$ where

$$\hat{Q}_A = \hat{A}\hat{Q}\hat{A}^{-1} = \hat{U}\hat{K}\hat{Q}(\hat{U}\hat{K})^{-1} = \hat{U}\hat{K}\hat{Q}\hat{K}^{-1}\hat{U}^{-1} = \hat{U}Q^*\hat{U}^\dagger \tag{4}$$

where $*$ denotes complex conjugation. Anti-unitary symmetry ($\hat{A}$) requires $\hat{Q}_A = \hat{Q}$. Anti-unitary operators do not possess eigenvalues; therefore the existence of $\hat{A}$ does not imply the partitioning of energy eigenstates into different symmetry classes.

The most familiar anti-unitary operator is the time-reversal operator $\hat{T}$. We adopt the conventional view that the effect of $\hat{T}$, classically and quantally, is to reverse all momenta. It then follows that in the position representation $\hat{U}=1$ in (1), i.e. $\hat{T}=\hat{K}$, while in the momentum representation $\hat{U}$ reverses the 'coordinate' p. This view is not quite correct, because $\hat{T}$ really corresponds to reversing *velocities*, which differ in a magnetic field from momenta by terms proportional to the vector potential (with a perverse choice of gauge this happens even in zero field). Nevertheless we adopt it because with a natural choice of gauge, which we discuss in § 3, invariance or lack of invariance of $\hat{H}$ under reversal of momenta really does imply the presence or absence of $\hat{T}$ respectively.

Suppose now that $\hat{H}$ has an anti-unitary symmetry. Then any basis $|\psi_n\rangle$ which is *A-adapted* in the sense that

$$\hat{A}|\psi_n\rangle=|\psi_n\rangle \tag{5}$$

will be a basis in which the matrix elements of $\hat{H}$ are real. The argument is very simple (Porter 1965, pp 2-87):

$$\begin{aligned}\langle\psi_m|\hat{H}|\psi_n\rangle&=\langle\hat{A}\psi_m|\hat{A}\hat{H}|\psi_n\rangle^* &&\text{(anti-unitarity)}\\&=\langle\hat{A}\psi_m|\hat{H}|\hat{A}\psi_n\rangle^* &&(\hat{A}\text{-symmetry})\\&=\langle\psi_m|\hat{H}|\psi_n\rangle^* &&(A\text{-adapted basis}).\end{aligned} \tag{6}$$

Does an A-adapted basis exist? We cannot prove this for every anti-unitary symmetry, but there are two important classes of $\hat{A}$ for which there is an abundance of A-adapted bases, indeed a continuous infinity of them.

The first class is that for which

$$\hat{A}^2=1 \tag{7}$$

and there is no restriction on the $\hat{U}$ in (1). Porter shows how to construct A-adapted bases where (7) holds; he was thinking of the case $\hat{A}=\hat{T}$, but we will apply his construction to a different example in § 4.

Our main concern will be with the second class, for which (7) need not hold but for which

$$\hat{U}^2=1 \tag{8}$$

(this does not imply (7) because $\hat{A}^2=\hat{U}\hat{K}\hat{U}\hat{K}=\hat{U}\hat{U}^*$ which in general equals neither $\hat{U}\hat{U}^+$ nor $\hat{U}^2$). Then $\hat{U}$ has eigenvalues $+1$ and -1, and correspondingly, eigenstates which are even or odd. To construct an A-adapted basis, the first step is to choose a representation (e.g. position) and take any complete set of functions $\{\phi_n\}$ with the property that in this representation they are all real. Next, we construct the states which are even or odd under $\hat{U}$ as follows:

$$\phi_n^+\equiv\phi_n+\hat{U}\phi_n, \qquad \phi_n^-\equiv\phi_n-\hat{U}\phi_n. \tag{9}$$

Thus

$$\hat{U}\phi_n^\pm=\pm\phi_n^\pm. \tag{10}$$

We assume that $\phi_n^\pm$ are real; this will be the case (in position representation) when $\hat{U}$ describes a geometric operation. Finally, we form the combinations

$$\psi_n^\pm\equiv\phi_n^+\pm\mathrm{i}\phi_n^-. \tag{11}$$

These functions form the A-adapted basis, because (1), (10) and the reality of $\phi_n^\pm$ give

$$\hat{A}\psi_n^\pm=\hat{U}\hat{K}\psi_n^\pm=\hat{U}\phi_n^+\mp\mathrm{i}\hat{U}\phi_n^-=\phi_n^+\pm\mathrm{i}\phi_n^-=\psi_n^\pm. \tag{12}$$

Obviously there is an infinity of these bases, corresponding to different choices of $\{\phi_n\}$. Moreover, as can easily be shown, transformation matrices between any two such bases are real, that is, orthogonal.

Thus, whenever a system is not invariant under T but is invariant under some other $\hat{A}$ in one of these two classes, we have a case of false T violation and predict that its spectrum will show GOE, rather than GUE, fluctuations.

3. Charged particle in a magnetic field: anticanonical symmetry

A particle with mass m and charge q moves in the $\boldsymbol{r}=(x,y)$ plane under the influence of a scalar potential $V(\boldsymbol{r})$ and a magnetic field $\boldsymbol{n}B(\boldsymbol{r})$ directed along the normal $\boldsymbol{n}$ to the plane. Its Hamiltonian, written classically but made quantal by replacing phase-space variables by operators, is

$$H(\boldsymbol{r},\boldsymbol{p})=(1/2m)\,(\boldsymbol{p}-q\boldsymbol{A}(\boldsymbol{r}))^2+V(\boldsymbol{r}) \tag{13}$$

where the vector potential $\boldsymbol{A}$ satisfies

$$\nabla\wedge\boldsymbol{A}(\boldsymbol{r})=\boldsymbol{n}B(\boldsymbol{r}). \tag{14}$$

To resolve the gauge ambiguity we choose $\boldsymbol{A}(\boldsymbol{r})$ to (i) vanish when B does, (ii) maximise the symmetry of $\boldsymbol{A}$ relative to B; this will lead to a unique gauge in which physical symmetries are not obscured. Firstly, we choose a Coulomb gauge in which $\nabla\cdot\boldsymbol{A}=0$ by representing $\boldsymbol{A}$ in terms of a scalar function $F(\boldsymbol{r})$, that is

$$A=\nabla\wedge(\boldsymbol{n}F(\boldsymbol{r})). \tag{15}$$

$\boldsymbol{A}$ is thus directed along the contours of F, which by (14) must satisfy

$$\nabla^2F(\boldsymbol{r})=-B(\boldsymbol{r}). \tag{16}$$

The maximally symmetric solution of this Poisson equation, involving the Green function with rotation symmetry, is

$$F(\boldsymbol{r})=\frac{1}{2\pi}\int\int \mathrm{d}^2\boldsymbol{r}'\,B(\boldsymbol{r}')\ln|\boldsymbol{r}-\boldsymbol{r}'|. \tag{17}$$

This choice of $F(\boldsymbol{r})$ vanishes when B does, and has the same symmetry as B. (Sometimes it is possible and convenient to choose a gauge uniquely related to $V(\boldsymbol{r})$, as explained by Berry and Robnik (1986) for the Aharonov-Bohm billiard, but this cannot be done in general and we do not consider it further here.)

With this gauge, the system has or does not have the physical symmetry T if H is or is not invariant under the $\hat{T}$ operation defined as $\boldsymbol{p}$ reversal. When $B=0$ then $\boldsymbol{A}=0$ and H is symmetric in $\boldsymbol{p}$ and of course motion without a magnetic field *is* physically T-invariant; and, when $B\neq0$ then $\boldsymbol{A}\neq0$, H is not symmetric in $\boldsymbol{p}$ and motion in a magnetic field is *never* T-invariant. This is true classically, even for B constant and V zero in which case velocity reversal results not in the particle retracing its Larmor circle but gyrating in the same sense round a different circle, and *a fortiori* in quantum mechanics.

Now we discuss three important symmetry operations and their consequences for the classical or quantal Hamiltonian (13). These considerations generalise our preliminary purely classical discussion (Robnik and Berry 1986). The operations are

$$\begin{aligned} &\text{Space inversion (parity)} P: && (x, y, p_x, p_y) \to (-x, -y, -p_x, -p_y) \\ &\text{Time reversal } T: && (x, y, p_x, p_y) \to (x, y, -p_x, -p_y) \\ &\text{Reflection } S_x: && (x, y, p_x, p_y) \to (-x, y, -p_x, p_y). \end{aligned} \qquad (18)$$

Whether H is invariant under these operations, or under combinations of them, depends on the symmetries of $B(\boldsymbol{r})$ and $V(\boldsymbol{r})$. The following four assertions follow easily from (13), (15) and (17):

(i) H has P if both V and B have P

(ii) H has T if $B=0$

(iii) H has S_x if $B=0$ and V has S_x. Note that if $B \neq 0$ and both B and V have S_x, then H does *not* have S_x; instead, we have the following example of false T violation:

(iv) H has TS_x if both V and B have S_x.

Quantally, the symmetries (i) and (iii) are unitary, with operators that we call $\hat{P}$ and $\hat{S}$. In contrast, (ii) and (iv) are anti-unitary, with operators $\hat{T}$ and $\hat{S}_x\hat{T}$. Note that $\hat{P}^2 = \hat{S}_x^2 = \hat{T}^2 = (\hat{S}_x\hat{T})^2 = 1$.

Classically, there is likewise an important distinction between (i), (iii) and (ii), (iv), which is the classical analogue of the distinction between unitary and anti-unitary symmetries. The phase-space transformations underlying (i) and (iii) are canonical; they leave invariant both H and Hamilton's equations, the Poisson brackets remaining unchanged. They correspond to quantal unitary operators. On the other hand the transformations underlying (ii) and (iv), while leaving H invariant, change the sign of Hamilton's equations and the Poisson brackets, these sign changes being equivalent to time reversal. We call these transformations *anticanonical*; they correspond to quantal anti-unitary operators.

The distinction can be expressed in another way: canonical symmetries estabish equivalence between forward orbits, whereas anticanonical symmetries relate a forward orbit to a backward orbit. This is illustrated in figure 1 in the case where B is uniform and V is a billiard potential, for the canonical symmetry P (figure 1(a)) and the anticanonical symmetry TS_x (figure 1(b)). As we shall see in § 5, this difference in the

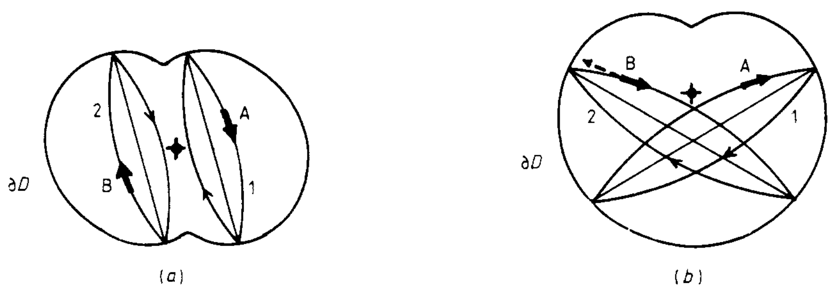


Figure 1. Pairs of classical orbits in billiards with boundary ∂D containing a uniform magnetic field. In (a) (canonical symmetry), ∂D has P but not S_x; P correctly moves point A on orbit 1 to point B on orbit 2 and reverses the velocity vector. In (b) (anticanonical symmetry), ∂D has S_x but not P; S_x correctly moves point A on orbit 1 to point B on orbit 2, but in reversing the x component of velocity (broken) gives an orbit traversed in the wrong sense; application of T corrects this by reversing the velocity vector at B, thus showing that the dynamics has the (false time-reversal violation) symmetry TS_x.

degeneracy structure of classical orbits has a striking effect on the distribution of quantal level spacings. Its effect on the spectral rigidity is much less marked (only a small asymptotic shift) because, according to the semiclassical theory (Berry 1985), degeneracy of closed orbits causes their contributions to add coherently, irrespective of the canonical or anticanonical nature of the symmetry responsible for the degeneracy.

There is a vast class of systems corresponding to the case (iv) of false T violation, where H has neither T nor S_x but does have TS_x. For all such systems we predict GOE spectral fluctuations provided the classical motion is chaotic. Important among these are atoms in strong uniform magnetic fields (Robnik (1981, 1982) has studied hydrogen), for which GOE statistics should apply to the subsets of levels with the same value of angular momentum about the symmetry axis, and the same symmetry under reflection about this axis and any perpendicular one. Another case of false T violation was found numerically by Seligman and Verbaarschot (1985) for a particle moving in smoothly inhomogeneous B and V. In the next section we present three examples designed to illustrate precisely the circumstances in which false T violation does or does not occur.

4. The Aharonov–Bohm quantum billiard

Following our recent work on this system (Berry and Robnik (1986) hereafter called ABQB), we now concentrate the magnetic field into a single line of magnetic flux Φ situated at what we define as the origin of the $\boldsymbol{r}$ plane, that is

$$B(\boldsymbol{r}) = \Phi\delta(\boldsymbol{r}), \tag{19}$$

and choose $V(\boldsymbol{r})$ to vanish within the planar domain D (billiard table) and be infinitely repulsive on and outside the boundary ∂D which is chosen to give chaotic classical trajectories. In polar coordinates $\boldsymbol{r} = (r, \theta)$, the vector potential is, from (15) and (17),

$$\boldsymbol{A}(\boldsymbol{r}) = (\Phi/2\pi r)\boldsymbol{u}_\theta \tag{20}$$

where $\boldsymbol{u}_\theta$ is the azimuthal unit vector.

The quantal Hamiltonian (13) is now entirely kinetic, with the effect of $V(\boldsymbol{r})$ being to make wavefunctions $\psi(\boldsymbol{r})$ vanish on ∂D. In addition, of course, ψ must be a single-valued function of $\boldsymbol{r}$. In terms of the *quantum flux parameter*

$$\alpha \equiv q\Phi/h \tag{21}$$

and apart from a factor $\hbar^2/2m$, the Hamiltonian operator in position representation is

$$\hat{H} = |\hat{\boldsymbol{p}}|^2 + \alpha^2/r^2 - 2\alpha\hat{l}/r^2. \tag{22}$$

In this expression, $\hat{\boldsymbol{p}}$ and $\hat{l}$ denote the linear and angular momentum operators, namely

$$\hat{\boldsymbol{p}} = -\mathrm{i}\nabla \qquad \hat{l} = -\mathrm{i}\,\partial/\partial\theta. \tag{23}$$

We now discuss five different cases depending on the value of α and the symmetry of ∂D.

4.1. $\alpha = 0$ and ∂D has no symmetry

Here there is no magnetic field, so the system has T and this is its only symmetry. The spectral statistics are those of the GOE and this has already been demonstrated in ABQB.

4.2. 2α is a non-zero integer and ∂D has no symmetry

For this case too, as we demonstrated in ABQB, there exists a transformation making $\hat{H}$ real and because non-zero α means non-zero field, this corresponds to false T violation. To explain this using the machinery of § 2, we first write (22) in the form

$$\hat{H} = \exp(\mathrm{i}\alpha\theta)|\hat{\boldsymbol{p}}|^2 \exp(-\mathrm{i}\alpha\theta). \tag{24}$$

Lack of T is obvious from (4), because the time-reversed operator is (with $\hat{T}$ as $\hat{K}$, i.e. $\hat{U} = 1$ in (1))

$$\hat{H}_T = \hat{H}^* = \exp(-\mathrm{i}\alpha\theta)|\hat{\boldsymbol{p}}|^2 \exp(\mathrm{i}\alpha\theta) \neq \hat{H}. \tag{25}$$

If we now choose the anti-unitary operator

$$\hat{A} = \exp(2\mathrm{i}\alpha\theta)\hat{K} \tag{26}$$

(which has the form (1)), the transformed Hamiltonian is, from (4),

$$\begin{aligned}\hat{H}_A &= \exp(2\mathrm{i}\alpha\theta)\hat{H}_T \exp(-2\mathrm{i}\alpha\theta)\\ &= \exp(2\mathrm{i}\alpha\theta)(\exp(-\mathrm{i}\alpha\theta)|\boldsymbol{p}|^2 \exp(\mathrm{i}\alpha\theta)) \exp(-2\mathrm{i}\alpha\theta)\\ &= \exp(\mathrm{i}\alpha\theta)|\boldsymbol{p}|^2 \exp(-\mathrm{i}\alpha\theta) = \hat{H}.\end{aligned} \tag{27}$$

We now seem to have proved too much, by finding an $\hat{A}$ under which $\hat{H}$ is invariant *for all* α! However, this ignores the fact that the specification of a Hamiltonian consists not only of the expression of the energy in terms of dynamical operators, but must include any conditions that the wavefunction $\psi(r, \theta)$ satisfies. Therefore, we must check that the action of $\hat{A}$ leaves these conditions invariant. Using (26) we obtain the transformed wavefunction as

$$\hat{A}\psi(r, \theta) = \exp(2\mathrm{i}\alpha\theta)\psi^*(r, \theta). \tag{28}$$

Considering now circuits of the flux line ($\theta \to \theta + 2\pi$), we see that $\hat{A}$ destroys the single-valuedness of ψ *unless* 2α *is an integer.* Only in this case does (26) lead to an anti-unitary symmetry of $\hat{H}$.

This case of false T violation, of dynamical rather than geometrical origin, corresponds to the first class considered in § 2, where (7) is satisfied but (8) is not. It is instructive to form an A-adapted basis by the method described by Porter (1965). We begin with the complete set of states

$$\phi_l = \exp(\mathrm{i}l\theta) \qquad (l \text{ any integer}) \tag{29}$$

(we write only the angle dependence), and then form the combination

$$\psi_l = a_l\phi_l + a_l^* \hat{A}\phi_l = a_l\exp(\mathrm{i}l\theta) + a_l^* \exp[\mathrm{i}(2\alpha - l)\theta] \qquad (l \geq 2\alpha) \tag{30}$$

(the restriction on l is necessary to avoid redundancy, and the loss of information resulting from the missing l values is compensated by the freedom of choice of the phases of the a_l). It is easy to show that $\hat{A}\psi_l = \psi_l$ so that we do indeed have an A-adapted basis, giving real matrix elements of $\hat{H}$ and hence GOE spectral statistics. The set $\{\psi_l\}$ is complete and orthonormal if 2α is an integer. One way to see how the functions (30) arise naturally is to write the complete wavefunction $\psi(\boldsymbol{r})$ in terms of the free-space Aharonov–Bohm eigenfunctions (Olariu and Popescu 1985), namely

$$\psi(\boldsymbol{r}) = \sum_{l=-\infty}^{\infty} c_l J_{|l-\alpha|}(kr) \exp(\mathrm{i}l\theta) \tag{31}$$

(where the J are Bessel functions and $k = [(2mE)^{1/2}]/\hbar$), and note that when 2α is an integer the terms l and $-l + 2\alpha$ have the same r dependence and can be amalgamated.

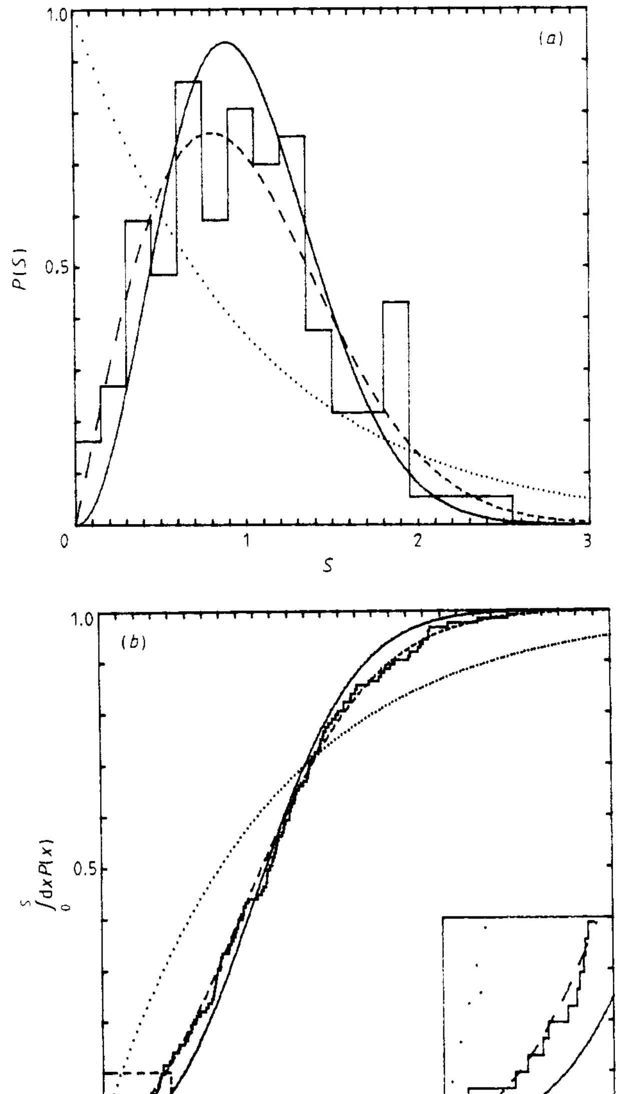


Figure 2.

4.3. 2α is not an integer and ∂D has no symmetry

Here there is neither unitary nor anti-unitary symmetry and the spectral statistics are those of the GUE, as we have already demonstrated in ABQB. This is a case of genuine T violation.

4.4. 2α is not an integer and ∂D has S_x

Here there is only the anti-unitary symmetry TS_x (case (iv) of § 3), indicating false T violation of geometric origin. Therefore, we predict GOE statistics. To demonstrate these we choose $\alpha = \frac{1}{2}(\sqrt{5}-1)$ (the golden flux) and ∂D as in figure 1(b); this is the

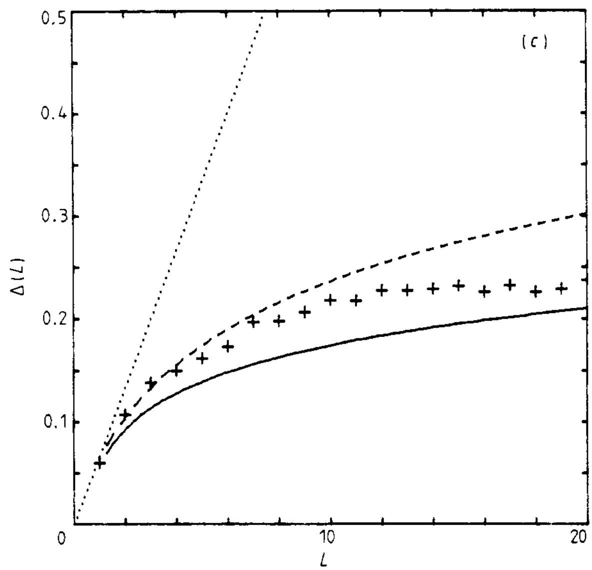


Figure 2. Spectral statistics for Aharonov-Bohm quantum billiard with golden flux $\alpha = \frac{1}{2}(\sqrt{5}-1)$ with the boundary ∂D given by (33) and shown in figure 1(b) (the flux passes through the marked point), which has the mirror symmetry S_x, but not P. (a) Level spacings distribution $P(S)$, (b) cumulative level spacings distribution $\int_0^S \mathrm{d}x P(x)$, ($c$) spectral rigidity $\Delta(L)$. The full, broken and dotted curves give the theoretical predictions of the GUE, GOE and Poisson statistics respectively.

image in the plane $z = x + \mathrm{i}y$ produced from the unit disc in the ζ plane by the quadratic conformal transformation

$$z(\zeta) = \mathrm{i}(\zeta + B\zeta^2) \qquad (B = 0.4). \tag{32}$$

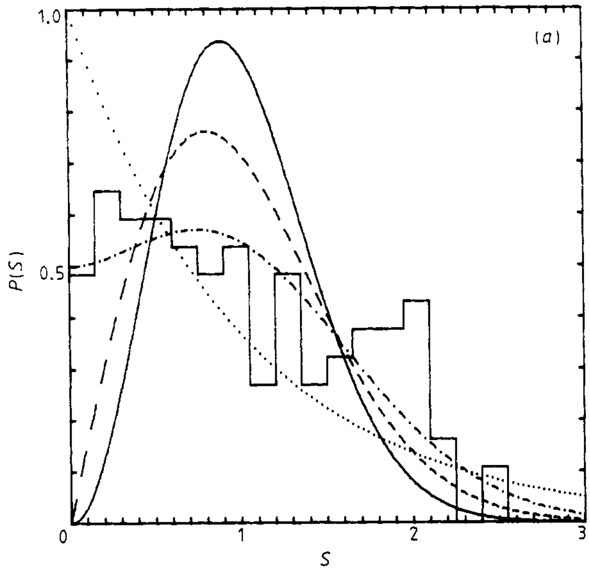
A complete description of our method of calculating eigenvalues with this boundary will be found in ABQB; also defined there are the spectral statistics we calculated with these eigenvalues, namely the level spacings distribution $P(S)$ and its integral $\int_0^S \mathrm{d}x\, P(x)$, and the spectral rigidity $\Delta(L)$.

Figure 2 shows the results. There is no doubt that the statistics are those of the GOE rather than the GUE (the deviation of $\Delta(L)$ from the GOE curve for large L is not unexpected because, as explained by Berry (1985), this lies beyond the range of spectral universality).

4.5. 2α is not an integer and ∂D has P as its only symmetry

Here, there is only the unitary symmetry P (case (i) of § 3) indicating a partition of the spectrum into two classes (even and odd under space inversion). For each separate class we predict GUE statistics because T is genuinely violated. For the complete spectral sequence, consisting of both classes of energy levels taken together, we predict the spectral statistics of two combined GUE sequences. To demonstrate these we again choose the golden flux $\alpha = \frac{1}{2}(\sqrt{5} - 1)$, but now take ∂D as in figure 1(a); this is the following quintic conformal transformation of the unit disc:

$$z(\zeta) = \zeta + C\zeta^3 + \mathrm{i}D\zeta^5 \qquad (C = 0.2, D = 0.05). \tag{33}$$



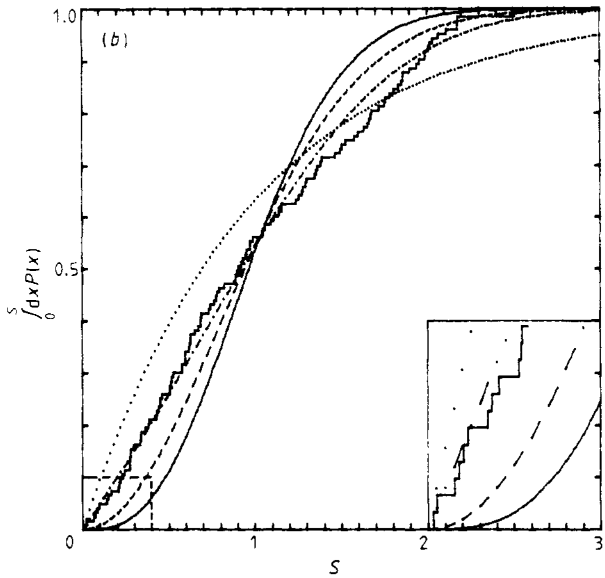


Figure 3.

The combination of two identical spectral sequences was first considered by Gurevich and Pevsner (1956/7) (reprinted in Porter 1965). Their arguments give, for the level spacings distribution, $P_{2GUE}(S)$, produced by combining sequences with level spacings distributions $P_{GUE}(S)$,

$$P_{2GUE}(S) = 2(d/dS)(\psi(S)\, d\psi(S)/dS) \tag{34}$$

where

$$\psi(S) = \tfrac{1}{2} \int_{S/2}^{\infty} dy(2y - S) P_{GUE}(y). \tag{35}$$

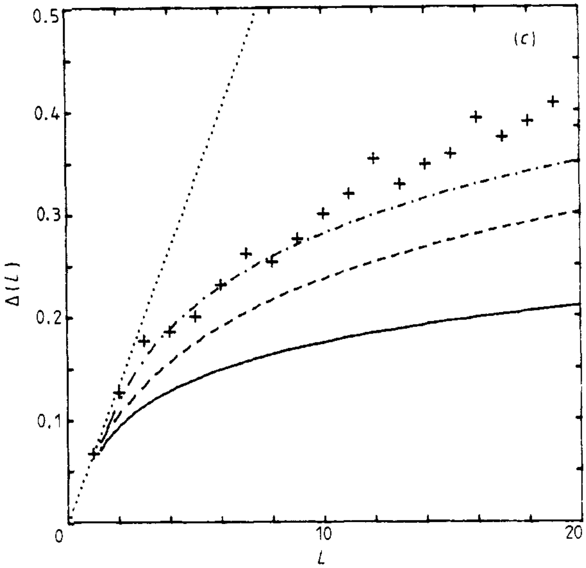


Figure 3. As figure 2 but with the boundary ∂D given by (34) and shown in figure 1(*a*), which has the parity symmetry P but not S_x. The chain curves give the theoretical predictions for two combined GUE spectra (equations (38) and (40)).

The close approximation

$$P_{\mathrm{GUE}}(S) \simeq (32/\pi^2) S^2 \exp(-4S^2/\pi) \tag{36}$$

leads to an expression for the combined cumulative level spacing in terms of the complementary error function

$$\int_0^S \mathrm{d}x\, P_{2\mathrm{GUE}}(x) = 1 - 2[(S/\pi)\exp(-S^2/\pi) + \tfrac{1}{2}\operatorname{erf}(S/\sqrt{\pi})] \times [\exp(-S^2/\pi) - (S/2)\operatorname{erf}(S/\sqrt{\pi})]. \tag{37}$$

Differentiation gives $P_{2\mathrm{GUE}}(S)$. In particular,

$$P(0) = \tfrac{1}{2}, \qquad \mathrm{d}P(0)/\mathrm{d}S = 0, \qquad \mathrm{d}^2P(0)/\mathrm{d}S^2 = 8/\pi^2. \tag{38}$$

Figures 3(*a*) and 3(*b*) show $P(S)$ and $\int_0^S \mathrm{d}x P(S)$ respectively. It is clear that (37) and its derivative provide excellent descriptions of the data over the whole range of S.

Because the rigidity is a quadratic functional of the level density, its value is simply additive for two independent spectral sequences. Thus

$$\Delta_{2\mathrm{GUE}}(L) = 2\Delta_{\mathrm{GUE}}(L/2). \tag{39}$$

The limiting forms

$$\begin{aligned} \Delta_{\mathrm{GUE}}(L) &\to L/15 && (L \ll 1) \\ &\to (1/2\pi^2)\ln L + 0.0590 && (L \gg 1) \end{aligned} \tag{40}$$

give

$$\begin{aligned} \Delta_{2\mathrm{GUE}}(L) &\rightarrow L/15 && (L \ll 1) \\ &\rightarrow (1/\pi^2)\ln L + 0.0478 && (L \gg 1). \end{aligned} \tag{41}$$

The rigidity data are shown in figure 3(*c*). Evidently the agreement with $\Delta_{2\mathrm{GUE}}$ (equation (39)) is excellent. In particular, the rigidity is clearly different from that of the GOE, which has the same logarithmic term but whose asymptotic constant is 0.0547 smaller.

In two dimensions, P is equivalent to symmetry under rotation by π. More generally, invariance under rotation by $2\pi/n$ (n integer) is a unitary symmetry, and if the potential has no reflection symmetry then in a magnetic field there will be neither T nor any other anti-unitary symmetry, the spectrum being a combination of n GUE sequences.

5. Discussion

The principal conclusion of this work is a warning: before predicting that the statistics of the GUE will apply to the energy levels of a classically chaotic system with neither time reversal nor geometric symmetry, make sure that the system is not invariant under some combination of these symmetries. If it is, real representations of the Hamiltonian can be found, indicating GOE statistics. Such cases correspond to invariance under the action of anti-unitary operators (classically, anticanonical transformations), which we have shown to be the appropriate generalisation of time reversal.

Finally, we must discuss the status of applications of random-matrix theory to predict the spectral statistics of individual systems, after all symmetries have been properly identified. We claim that such predictions, based on whether the Hamiltonian is real or complex, will be correct with probability one, but not with certainty.

It is easy to give examples proving that the predictions will not always be correct. Observe that the Hamiltonian matrix of any system is real in its eigenbasis, whether the system has GOE, GUE or any other spectral statistics. Applying this to the case when the levels have GUE statistics, we can transform the eigenbasis by arbitrary orthogonal transformations into a continuous infinity of representations, in all of which the Hamiltonian is real; all these real matrices have the same, GUE, statistics. Conversely, one may take a real Hamiltonian whose levels have GOE statistics and make arbitrary complex unitary transformations, thereby producing a continuous infinity of representations, in all of which the Hamiltonian is complex; all these complex matrices have the same, GOE, statistics. An alternative procedure for generating 'wrong' fluctuations is to begin with any Hamiltonian $\hat{H}_0$ and deform its spectrum by mapping it onto that of a new Hamiltonian $\hat{H} = f(\hat{H}_0)$; if f is a sufficiently complicated (but smooth) function, any desired spectral statistics can be produced. For example, it has been suggested (Robnik 1985) that it could even be possible to construct integrable systems whose spectral statistics mimic those of chaotic ones; such special integrable systems would be obtained from truncated Birkhoff–Gustavson normal forms.

However, these counterexamples to random-matrix predictions, although infinitely numerous, are non-generic: they constitute a set of measure zero. We know this from random-matrix theory, as will now be explained. The matrices with 'wrong' statistics are all included in the Gaussian ensembles: the GOE includes all those real matrices which have, individually, GUE statistics, and the GUE includes all those complex matrices which have, individually, GOE statistics (as well as all the real symmetric matrices of the GOE). However, for infinite matrices, the Gaussian ensembles are *ergodic* (Pandey 1979): almost every individual matrix in each of these has the same statistics as the average over the whole ensemble. Therefore the exceptions do indeed have zero measure.

Confidence in random-matrix predictions is further strengthened in this context of classically chaotic quantum systems by the realisation that spectral statistics must be calculated, in principle, using infinitely many levels, which for an individual system implies that the statistics are dominated by the *semiclassical limit*. For systems such as billiards (field-free or Aharonov–Bohm with fixed α) the energy and $\hbar$ are related by scaling, and the semiclassical limit simply corresponds to fixing $\hbar$ and considering the whole spectrum. For non-scaling systems, where the classical dynamics is energy dependent, the statistics of levels corresponding to given classical mechanics at some fixed energy E must be calculated by choosing a fixed small energy range ΔE, centred on E, and letting $\hbar \to 0$, thus causing infinitely many levels to condense into this range. This eliminates the possibility of producing GOE or GUE statistics by a smooth $\hbar$-independent mapping $\hat{H} = f(\hat{H}_0)$, because in the semiclassical limit, the transformation f becomes locally a simple multiplication by the constant factor $\mathrm{d}f/\mathrm{d}E$ and this does not affect the local structure of the spectrum.

We conclude that failures of predictions of spectral statistics of classically chaotic quantum systems on the basis of random-matrix theory will be infinitely unlikely, provided all symmetries are correctly taken into account. This common-sense view is of course confirmed not only by the numerical calculations reported here, but also by computations carried out by us, and other people, for a variety of different systems.

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